A CONCEPT OF MASS ENTRAINMENT APPLIED TO COMPRESSIBLE TURBULENT BOUNDARY LAYERS IN ADVERSE PRESSURE GRADIENTS

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ABSTRACT

The problem of determining the growth of a turbulent boundary layer under conditions occurring on ramjet air-intakes is discussed in the introduction. Under "analysis," the compressible integral momentum equation is transformed to an equivalent incompressible plane, adopting approximate temperature-velocity relations of the Crocco and van Driest form. A semiempirical auxiliary equation, developed by M. R. Head for incompressible flow and using the concept of mass entrainment into the boundary layer, is rearranged for use in the transformed plane. The theoretical results obtained by this method are then compared to McLafferty's lag-length theory, and to experimental data obtained by C. E. Kepler and R. L. O'Brien. It is seen that the mass-entrainment theory is in good qualitative agreement with the reported data, and in reasonable quantitative agreement. The latter may be improved by postulating a relationship between temperature and velocity in the boundary layer which is in closer agreement with experiment. It appears that the mass-entrainment theory indicates the point of separation of the compressible, turbulent boundary layer, in accordance with conventional incompressible separation criteria.

INTRODUCTION

The study of turbulent boundary layers is generally recognized to be a task of more than slight complexity, for the random eddying motion of the turbulent fluid is not amenable to simple mathematical description. In addition, such phenomena as shearing stresses and heat transfer across a turbulent layer are no longer proportional to parameters which are properties of the fluid alone, as in the case of laminar flows.

Additional complexities, such as compressibility of the fluid, adverse pressure gradients in the flow direction, heat transfer between the fluid and its bounding surface, centrifugal forces acting on the fluid and surface roughness are often engineering realities which cannot be ignored. An example of the necessity of knowing accurately the behaviour of boundary layers under these and other conditions is found in the study of supersonic combustion ramjets. The inlet diffuser of such a ramjet is a curved surface over which flows a compressible fluid under an adverse pressure gradient. Due to the high recovery temperatures of high-speed flight, the inlet surface must necessarily be cooled, thus causing heat transfer from the fluid to the surface. These are the basic conditions experienced by a turbulent boundary layer on a ramjet diffuser, and consequencly comprise the parameters of this analysis.

ANALYSIS

The shear stress distribution is not known analytically for turbulent motion. Therefore the averaged or integral method is widely used in analyzing turbulent boundary layers, since a knowledge of the shear stress variation is not required in this method. Calculation procedures using integral methods for the problem of compressible turbulent boundary layers under the effects of pressure gradients and heat transfer have been suggested by a great number of workers, among them Reshotko and Tucker, McLafferty and Barber, Stroud and Coleman, and Sivells and Payne [5,7,8,4].

In the following analysis an ideal gas has been assumed, with a specific heat ratio of 1.4. It was felt that the ideal gas characteristics were sufficiently accurate in the thermodynamic regime covered by the experimental results, and that allowances for real gas effects would therefore needlessly complicate the analysis. Some real gas effects may be included without too much alteration or difficulty.

MOMENTUM EQUATION

The integral momentum equation for compressible turbulent flow is obtained by integrating the Prandtl momentum equation across the boundary layer thickness [9]. If the turbulent flow properties are represented as the sum of time-mean values and fluctuating components, then certain terms involving the fluctuating components appear in the integral momentum equation [6]. These terms usually may be neglected except in the region of separation or in the presence of large centrifugal forces acting on the fluid [1,6]. In Ref. 1 it is observed that these fluctuation terms, which include the variation of static pressure in the direction normal to a curved compression surface, may be neglected for moderately curved surfaces. Thus, the compressible integral momentum equation becomes

$$\frac{d\theta}{dx} + \frac{\theta}{M_e}\frac{dM_e}{dx}\left(\frac{2 + H - M_e^2}{T_0/T_e}\right) = \frac{C_f}{2} \tag{2*}$$

Normal pressure gradients due to centrifugal forces may be neglected on compression surfaces where the radii of curvature are large in comparison with the boundary layer thickness.

The skin friction coefficient C_f must also be defined for turbulent flow. In the present analysis, the Ludwig-Tillman equation for incompressible flow is used, with fluid properties evaluated at Eckert's reference temperature, following the procedure of Ref. 5. The resulting expression employing Sutherland's law of viscosity is

$$\frac{C_f}{2} = 0.123 \exp\,(-1.56\,H_1)\,(U_e\theta_i\rho_0/\mu_0)^{-0.268}\,\frac{T_e}{T_r}\left(\frac{T_r}{T_0}\right)^{0.402}\left(\frac{T_0 + 198}{T_r + 198}\right)^{0.268} \tag{3}$$

and is substituted into Eq. (2) in this form.

The momentum equation is now transformed to a form similar to the integral momentum equation for incompressible flow. A modified Stewartson transformation is used. Defining the transformation for the normal coordinate by

$$dY = \frac{\rho a_e}{\rho_0 a_0}\,dy \tag{4}$$

and by equating the compressible stream function to the transformed stream function, the result

$$U = \frac{a_0}{a_e}\,u \tag{5}$$

is obtained.

Under the transformation, the velocity ratio u/u_e is equal to the ratio of transformed velocities U/U_e.

*There is no Eq. (1) [Ed.]

Employing the Stewartson transformation to the integral boundary layer quantities, then, and recalling that the static pressure remains constant through the boundary layer, the transformed integral parameters are defined [4] as

$$\theta_i = \left(\frac{T_e}{T_0}\right)^3 \theta$$

$$\delta^*_{\mathrm{tr}} = \int_0^\Delta \left(\frac{T_s}{T_0} - \frac{U}{U_e}\right) dY \tag{6}$$

(See App. A)

$$H_{\mathrm{tr}} = \frac{\delta^*_{\mathrm{tr}}}{\theta_i}; \qquad H = \frac{T_0}{T_e} H_{\mathrm{tr}} + \frac{T_0}{T_e} - 1$$

Substituting Eqs. (6) into the compressible momentum equation (2), the transformed momentum equation is obtained

$$\frac{d\theta_i}{dx} + \frac{\theta_i}{M_e}\frac{dM_e}{dx}(2 + H_{\mathrm{tr}}) = \frac{C_f}{2}\left(\frac{T_e}{T_0}\right)^3 \tag{7}$$

This transformed equation is still not of the form of the incompressible momentum equation, however, since the transformation of the longitudinal coordinate "x" is undefined, and since the friction coefficient term is not equivalent to the incompressible skin friction. In addition, the transformed shape factor must be related to an equivalent incompressible shape factor in order to determine its variation.

TEMPERATURE DISTRIBUTION

A relationship between temperature and velocity at any point in the boundary layer is now required in order to relate the transformed shape factor to an equivalent incompressible shape factor (Appendix B).

Such a temperature-velocity relationship was obtained by Crocco as an exact solution of the momentum and energy equations under zero pressure gradient, assuming a Prandtl number of unity. Van Driest also obtained a similar relationship for a non-unit Prandtl number, although he assumed that the thermal boundary layer was the same thickness as the velocity layer. The first of these analyses resulted in a linear relationship between temperature and velocity, of the form

$$\frac{T_s}{T_0} = a + b\,\frac{u}{u_e} \tag{8}$$

with a and b constant.

Van Driest's equation was a quadratic in velocity ratio,

$$\frac{T_s}{T_0} = a + b\frac{u}{u_e} + c\left(\frac{u}{u_e}\right)^2 \tag{9}$$

The second derivative of this relation is positive, after evaluation of the constants from the boundary conditions.

However, under adverse pressure gradients, temperature-velocity curves obtained from experiments [1] exhibit a negative second derivative. It appears that the temperature distribution is at least of second order in velocity ratio, and has coefficients which yield negative second derivatives when the pressure gradient is considered.

The difficulty in obtaining an equation for the temperature distribution suggests the use of a simpler, although less accurate approximation. Consequently, both the Crocco and van Driest temperature distributions were used. Since the temperature distribution appears in integral or averaged relations only, the resulting error is not too severe.

AUXILIARY EQUATION

In either the compressible or transformed momentum equations, it is still necessary to evaluate the shape parameter H. Since this shape factor varies with the growth of the boundary layer, an equation defining its variation must be obtained.

Such an equation is empirical, however, since the physical concepts of conservation of momentum, energy, and mass do not yield a relation involving the variation of shape factor. Consequently, many different empirical or semiempirical equations have been suggested [4,5,7,9].

A concept of the rate of entrainment of external flow into the incompressible turbulent boundary layer, suggested by M. R. Head [2], has led to the formulation of another auxiliary equation for the shape factor variation. Head's auxiliary equation is a more promising approach to the problem since it involves the investigation of a physical phenomenon which is the basis of boundary-layer growth.

Head himself has demonstrated that the present form of his auxiliary equation yields relations which are very nearly the same as other forms of the auxiliary equation. Yet since Head's form of the equation is obtained from a consideration of the physics of the boundary layer, it merits closer attention and study in the future.

In his derivation [2] Head assumes that the rate of entrainment into a turbulent boundary layer depends upon a boundary-layer thickness parameter, the free stream velocity and the velocity distribution in the outer portion of the layer. Using nondimensional terms, Head arrived at a form of the auxiliary equation

$$\frac{d}{dX}(\Delta - \Delta^*) = F - (\Delta - \Delta^*)/M_e \frac{dM_e}{dX} \tag{10}$$

Using the experimental results of several papers, Head obtained an empirical correlation between the function F and the shape factor $H_{\Delta - \Delta^*} = (\Delta - \Delta^*)/\theta_i$, and a correlation between $H_{\Delta - \Delta^*}$ and H_i, Figs. 5 and 6. It should be emphasized that these results were obtained for incompressible flow.

SOLUTION OF THE EQUATIONS

The transformation of the momentum equation is completed by defining the x-coordinate transformation as

$$\frac{dX}{dx} = \frac{T_e}{T_r}\left(\frac{T_e}{T_0}\right)^3 \left(\frac{\mu_r}{\mu_0}\right)^{0.268} \qquad \text{(Ref. 3 and Appendix C)}$$

and by relating the transformed and equivalent incompressible shape factors (Appendix B).

Head's auxiliary equation, Eq. (10), is put into workable form by fitting equations to the curves in Figs. 5 and 6. The equations so obtained are

$$H_{\Delta-\Delta^*} = 1.535\,(H_i - 0.7)^{-2.715} + 3.3 \tag{11}$$

corresponding to Fig. 6, and

$$F = 0.0306\,(H_{\Delta-\Delta^*} - 3.0)^{-0.653} \tag{12}$$

corresponding to Fig. 5.

The resulting form of the auxiliary equation is

$$\frac{dH_i}{dx} = -\frac{(H_i - 0.7)^{3.715}}{4.17}\left(\frac{F}{\theta_i}\frac{dX}{dx} - \frac{H_{\Delta-\Delta^*}}{M_e}\frac{dM_e}{dx} - \frac{H_{\Delta-\Delta^*}}{\theta_i}\frac{d\theta_i}{dx}\right) \tag{13}$$

where $H_{\Delta - \Delta^*}$ and F are given by Eqs. (11) and (12). The derivation of this form is presented in Appendix C.

The momentum and auxiliary equations were solved using a simultaneous numerical solution of the Runge-Kutta type. The integral parameters so computed were then transformed to the compressible plane by means of Eq. (6).

COMPARISON WITH EXPERIMENT

A comparison of the calculation results with experiment will be limited to the one set of data obtained by Kepler and O'Brien [1] on a Mach 6 isentropic compression ramp at two wall temperatures. Although this data appear to be quite precise, any conslusions which are drawn on the basis of such a limited comparison must necessarily be speculative in nature.

An examination of the velocity profiles reported by Kepler and O'Brien, Figs. 1 and 2, indicates a full profile for both the uncooled and cooled surfaces at the initiation of compression, and inflected profiles for both wall temperatures in the final stages of compression.

Using such profiles in conjunction with total temperature profiles obtained by Kepler and O'Brien at five stations on the compression surface, temperature-velocity curves were plotted. Examples of these are given in Figs. 3 and 4. It was observed that the van Driest temperature-velocity relation provided good agreement with the experimental points over more than half of the compression surface for the uncooled wall condition. In the case of the cooled wall, neither the van Driest nor the Crocco relation lay among the experimental points, but the Crocco distribution was the nearer of the two. Consequently, in accordance with the argument set forth in the section "Analysis," the van Driest relation was used for the calculation of the uncooled boundary layer, and the Crocco equation for the cooled layer.

The calculation results are shown in Figs. 7 and 8. These curves indicate a better qualitative than quantitative agreement with experiment, although the uncooled wall results are in reasonable proximity to the experimental points. This is believed to be due to the closer agreement between the van Driest temperature distribution and experiment in the uncooled wall case, than between the Crocco relation and experiment in the cooled boundary layer. As a check on this assumption, the van Driest relation, giving even poorer agreement with experiment than the Crocco equation (Fig. 4) was used in the cooled wall calculation. The values of both the momentum thickness and displacement thickness were found to be lower at all points on the compression surface than those obtained using the Crocco relation. Although the difference was not great (of the order of 5 per cent) it was significant enough to indicate that a closer approximation to the true temperature-velocity relationship would yield better values of the integral parameters.

The calculation of the integral parameters using McLafferty's lag-length procedure was performed by Kepler and O'Brien, and is shown in Figs. 7 and 8. It is noted that the results of the lag-length theory indicate no increase in either the momentum thickness or displacement thickness in

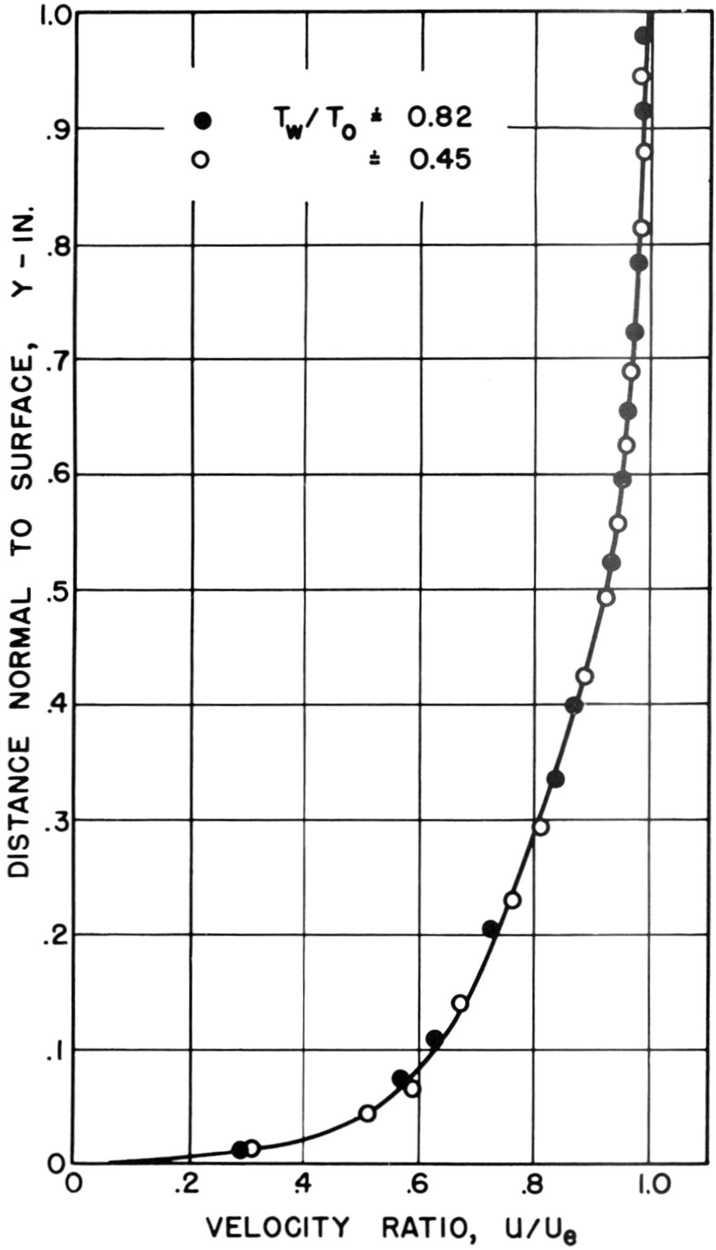

Figure 1. Velocity profiles at start of compression, from Ref. 1.

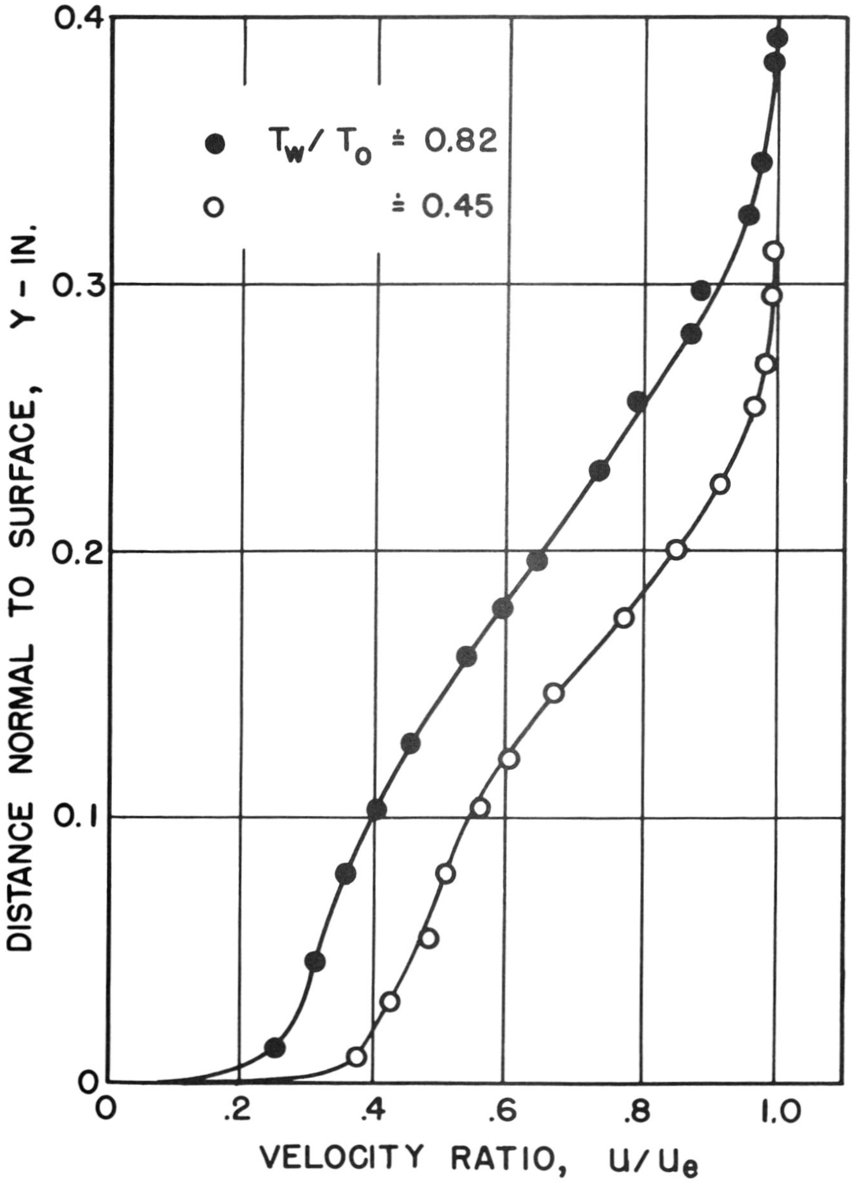

Figure 2. Velocity profiles in final stages of compression, from Ref. 1.

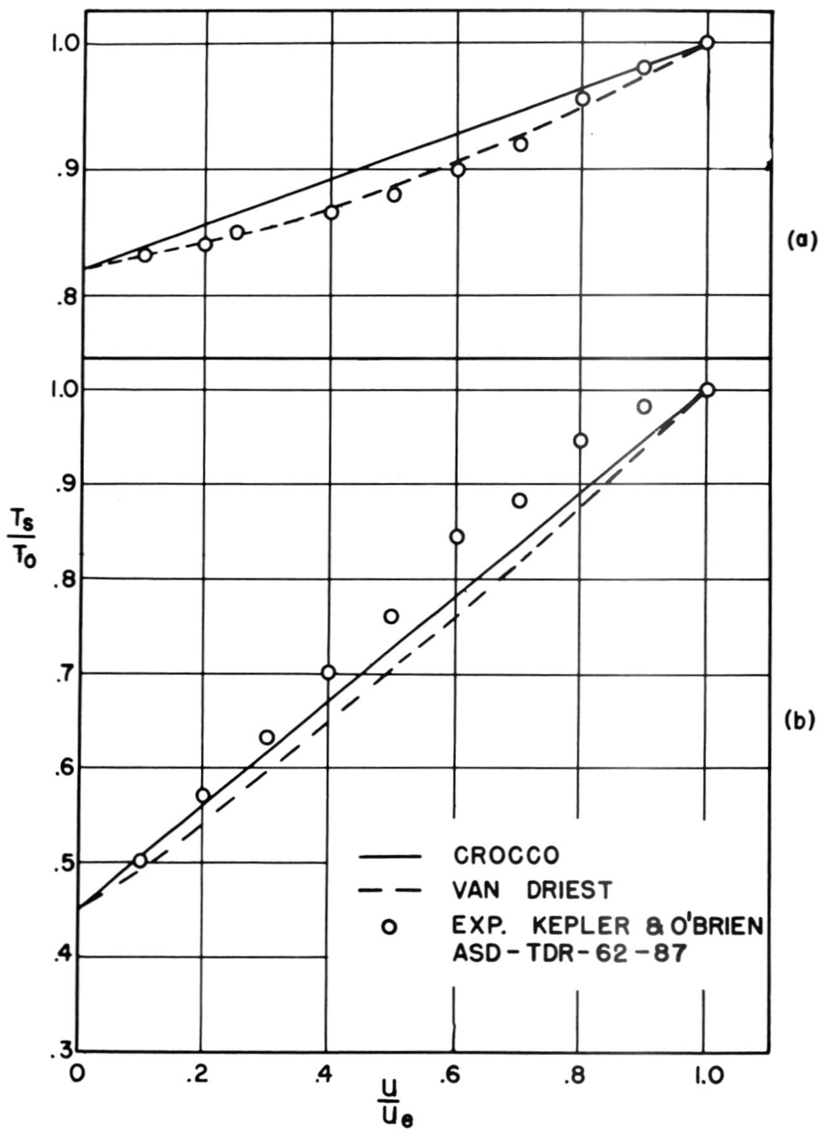

Figure 3. Temperature-velocity relationships, at start of compression, isothermal wall. (a) $T_w/T_0 \doteq 0.82$; (b) $T_w/T_0 \doteq 0.45$.

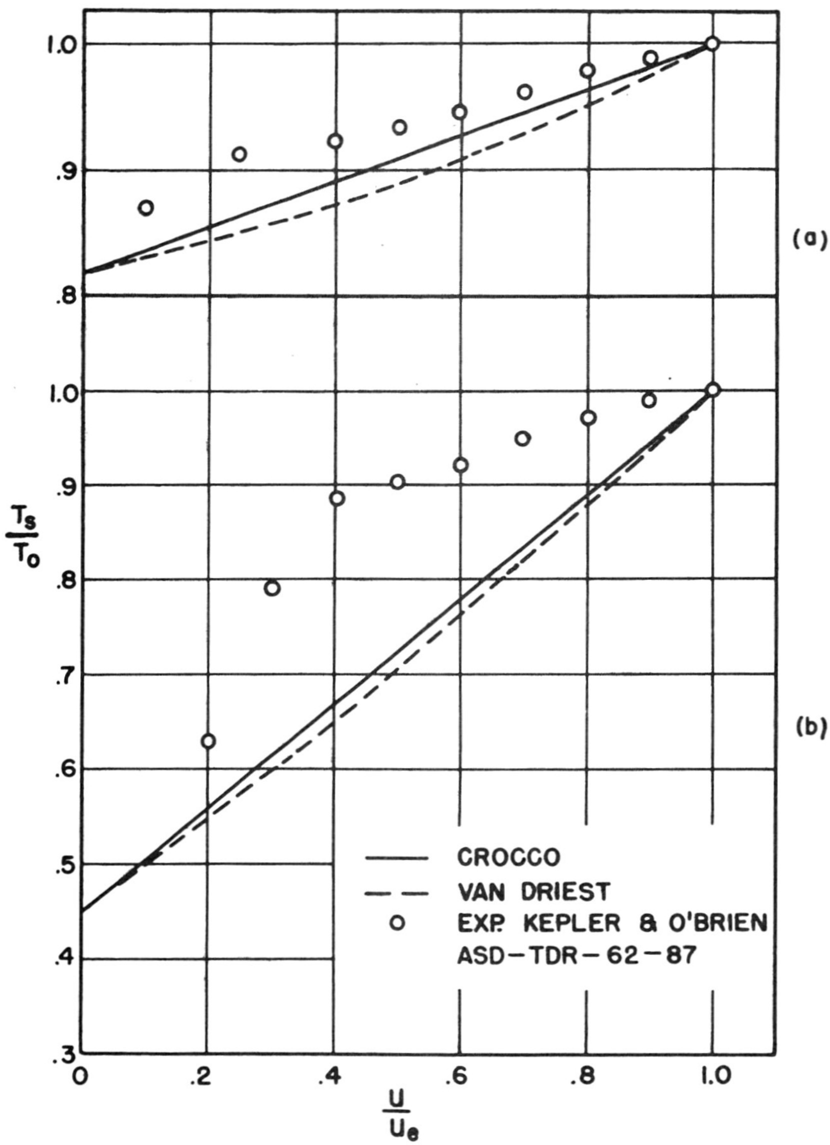

Figure 4. Temperature-velocity relationships in final stages of compression, isothermal wall. (a) $T_w/T_0 \doteq 0.82$; (b) $T_w/T_0 \doteq 0.45$.

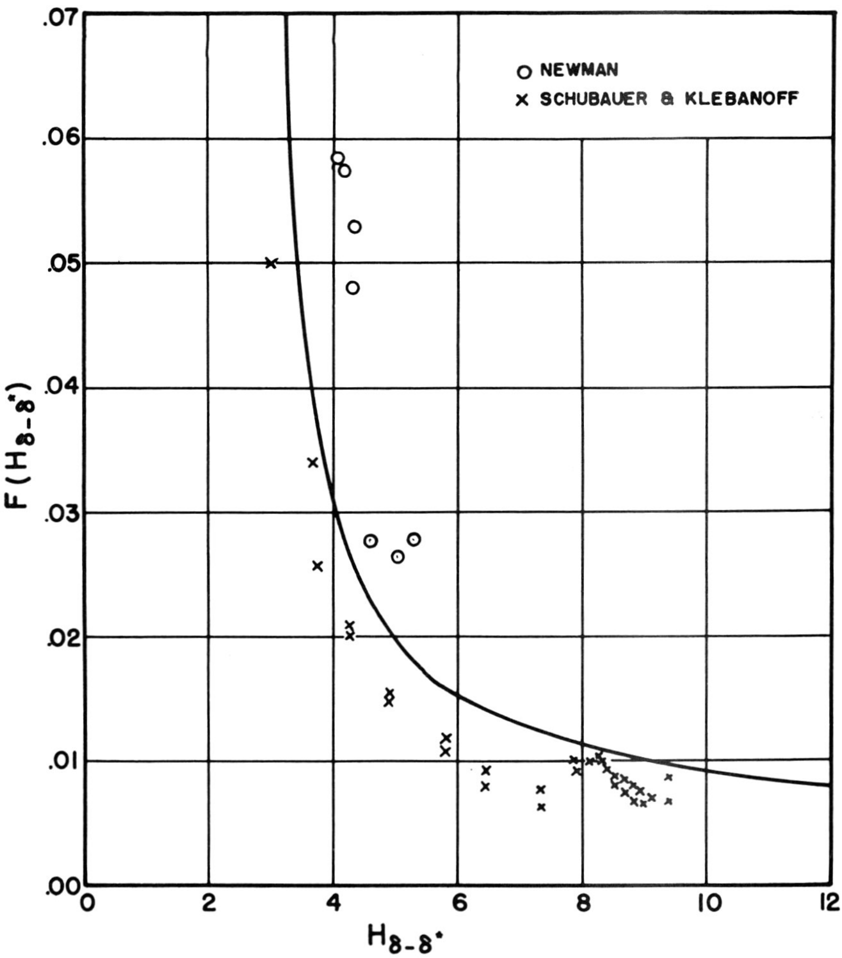

Figure 5. Correlation of functions F and $H_{\Delta - \Delta^*}$, from Ref. 2.

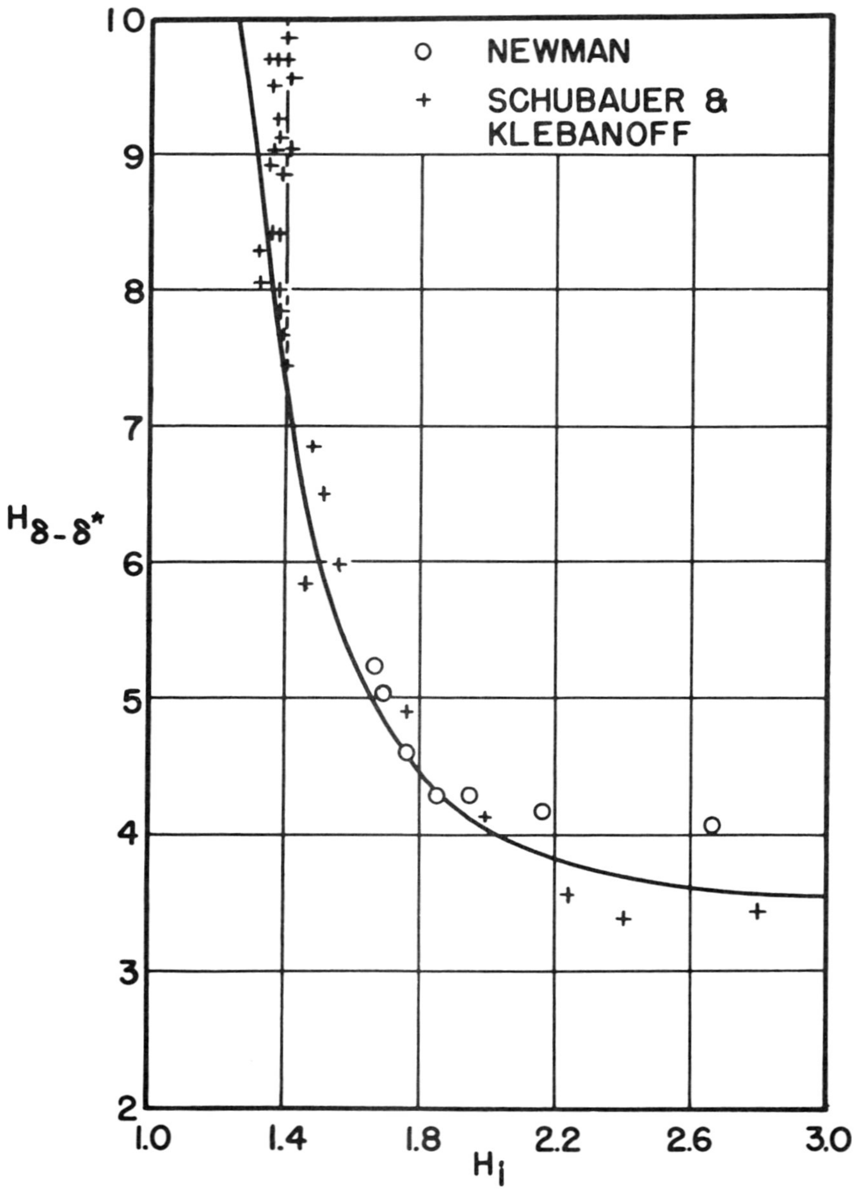

Figure 6. Correlation of shape factors $H_{\Delta - \Delta^*}$ and H_i, from Ref. 2.

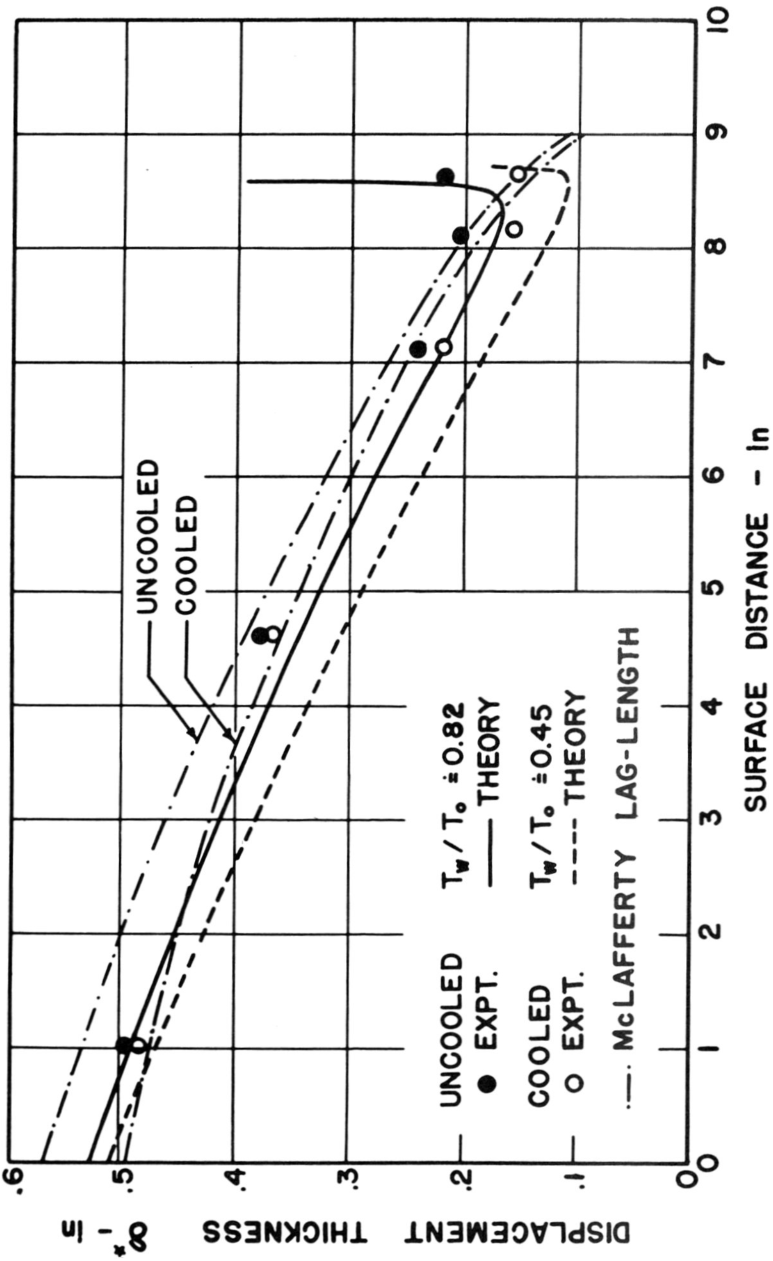

Figure 7. Variation of displacement thickness on isentropic compression surface of Ref. 1.

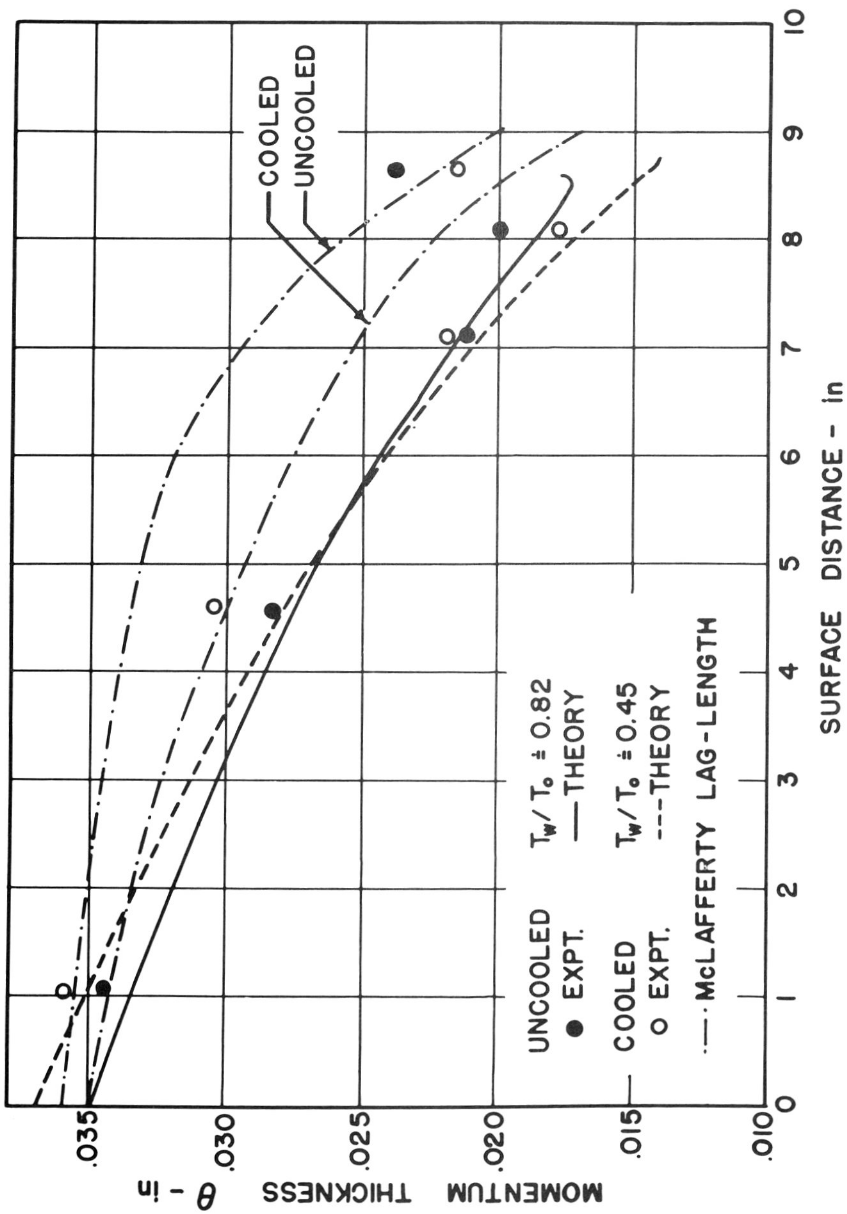

Figure 8. Variation of momentum thickness on isentropic compression surface of Ref. 1.

the final stages of compression, as is shown by the experimental data. The mass-entrainment method, however, does indicate such an increase, or at least a levelling-off, of the integral values in this region. The value of the equivalent incompressible shape factor, H_i, was between 2.4 and 5.0 at this point, and was increasing rapidly. Such behaviour of the shape factor in incompressible flow is generally accepted as a criterion of incipient separation. Although Kepler and O'Brien reported no occurrence of separation in their experiment [1], the existence of inflected profiles and the increase of integral values suggest that separation was approaching. As was pointed out in Ref. 1, the expansion of the flow at the end of the compression surface, occurring when the flow was returned to its original direction, could well have influenced the behaviour of the boundary layer in this region. The influence would be exhibited mainly in the subsonic portion of the boundary layer near the wall, and would have the effect of relaxing the inflected velocity profile, thus discouraging separation.

The calculation of the boundary layer presented here is a first order consideration. The addition of the displacement thickness to the surface profile would modify the Mach number distribution, especially near the end of compression where the displacement thickness increases rapidly. In this region, the modification would result in a locally increased adverse pressure gradient. One would expect that this result would tend to increase the values of the integral parameters in this area.

CONCLUSIONS

The conclusions reached in the preceding discussion may be summarized as follows:

1. The concept of mass entrainment by a turbulent boundary layer appears to provide the basis of a suitable auxiliary equation for calculation of the shape parameter H of the boundary layer. Empirical relations describing such an entrainment in incompressible flow are applicable to compressible flows as well, through a suitably defined mathematical transformation. The good qualitative agreement between experiment and theory obtained through this concept suggests that the entrainment relationship should be investigated further, and established on a better theoretical and mathematical basis.
2. Better quantitative theoretical results may be expected if a temperature-velocity relationship, providing closer agreement with experiment than the Crocco or van Driest form, is used. This implies

the inclusion of pressure gradient and heat transfer effects in such a relationship.

3. The mass entrainment method indicates separation of the boundary layer in a region where inflected velocity profiles and increasing values of integral parameters were observed in experiment. Separation was indicated by the behaviour of the incompressible shape factor H_i, in accordance with the usual criteria for incompressible flow. Further comparison with experiments must be undertaken before the capability of this method to predict incipient separation is established.

SYMBOLS

a, b, c	constant coefficients
a_e, a_0	sound speed at temperatures T_e and T_0 respectively
C_f	local skin friction coefficient
F	nondimensional mass entrainment rate, Ref. 2
H	compressible shape factor
H_{tr}	transformed shape factor
H_i	equivalent incompressible shape factor
$H_{\Delta - \Delta^*}$	shape factor associated with entrainment rate, Ref. 2
$q = \rho_e u_e^2$	
M_e	Mach number at outer edge of boundary layer
T	static temperature at height y in boundary layer
T_s	total temperature at height y in boundary layer
T_e	static temperature at outer edge of boundary layer
T_0	total temperature of external flow
T_{aw}	adiabatic wall temperature
T_w	wall temperature
T_r	reference temperature
u	compressible longitudinal velocity
U	transformed longitudinal velocity
x	longitudinal coordinate parallel to wall
X	transformed longitudinal coordinate
y	coordinate normal to wall
Y	transformed normal coordinate
γ	specific heat ratio
δ	boundary-layer thickness
δ^*	compressible displacement thickness
δ^*_{tr}	transformed displacement thickness
δ^*_i	incompressible displacement thickness

θ	compressible momentum thickness
θ_i	transformed or incompressible momentum thickness
μ	absolute viscosity
ρ	density
Δ	transformed boundary-layer thickness

SUBSCRIPTS

e	outer edge of boundary layer
0	evaluated at total temperature
r	evaluated at reference temperature
i	incompressible

APPENDIX A

DERIVATION OF TRANSFORMED INTEGRAL PARAMETERS USING STEWARTSON'S TRANSFORMATION

(a) Momentum Thickness

By definition

$$\theta = \int_0^\delta \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$$

Substituting Eq. (4), and recalling that $u/u_e = U/U_e$, we have

$$\theta = \frac{\rho_0 a_0}{\rho_e a_e}\int_0^\Delta \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) dY$$

where Δ is the transformed boundary-layer thickness. With $\gamma = 1.4$, this becomes

$$\theta = \left(\frac{T_0}{T_e}\right)^3 \theta_i \tag{A1}$$

where

$$\theta_i = \int_0^\Delta \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) dY \tag{A2}$$

so

$$\theta_i = \theta\left(\frac{T_e}{T_0}\right)^3 \tag{A3}$$

(b) Displacement Thickness

By definition

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$

or

$$\delta^* = \int_0^\delta \frac{\rho}{\rho_e}\left(\frac{\rho_e}{\rho} - \frac{u}{u_e}\right) dy$$

Again using Eq. (4), we obtain

$$\delta^* = \frac{\rho_0 a_0}{\rho_e a_e} \int_0^\Delta \left(\frac{\rho_e}{\rho} - \frac{U}{U_e}\right) dY$$

By assuming constant static pressure normal to the wall,

$$\frac{\rho_e}{\rho} = \frac{T}{T_e}$$

and

$$\delta^* = \left(\frac{T_0}{T_e}\right)^3 \int_0^\Delta \left(\frac{T}{T_e} - \frac{U}{U_e}\right) dY$$

Now

$$\frac{T}{T_e} = \frac{T_0}{T_e}\left[\frac{T_s}{T_0} - \frac{u_e^2}{2c_pT_0}\left(\frac{u}{u_e}\right)^2\right]$$

$$= \frac{T_0}{T_e}\left[\frac{T_s}{T_0} - \left(1 - \frac{T_e}{T_0}\right)\left(\frac{U}{U_e}\right)^2\right]$$

So

$$\frac{T}{T_e} - \frac{U}{U_e} = \frac{T_0}{T_e}\frac{T_s}{T_0} - \left(\frac{T_0}{T_e} - 1\right)\left(\frac{U}{U_e}\right)^2 - \frac{U}{U_e}$$

$$= \frac{T_0}{T_e}\frac{T_s}{T_0} - \frac{T_0}{T_e}\frac{U}{U_e} + \frac{T_0}{T_e}\frac{U}{U_e} - \left(\frac{T_0}{T_e} - 1\right)\left(\frac{U}{U_e}\right)^2 - \frac{U}{U_e}$$

Gathering terms

$$\frac{T}{T_e} - \frac{U}{U_e} = \frac{T_0}{T_e}\left(\frac{T_s}{T_0} - \frac{U}{U_e}\right) + \left(\frac{T_0}{T_e} - 1\right)\frac{U}{U_e}\left(1 - \frac{U}{U_e}\right)$$

then

$$\delta^* = \left(\frac{T_0}{T_e}\right)^3 \int_0^\Delta \left[\frac{T_0}{T_e}\left(\frac{T_s}{T_0} - \frac{U}{U_e}\right) + \left(\frac{T_0}{T_e} - 1\right)\frac{U}{U_e}\left(1 - \frac{U}{U_e}\right)\right] dY$$

or

$$\delta^* = \left(\frac{T_0}{T_e}\right)^3 \left[\frac{T_0}{T_e}\,\delta^*_{\mathrm{tr}} + \left(\frac{T_0}{T_e} - 1\right)\theta_i\right] \tag{A4}$$

using Eq. (A2) and defining

$$\delta^*_{\mathrm{tr}} = \int_0^\Delta \left(\frac{T_s}{T_0} - \frac{U}{U_e}\right) dY \tag{A5}$$

Now, using Eq. (A4) and (A1), we obtain

$$H = \frac{\delta^*}{\theta} = \frac{T_0}{T_e} H_{\mathrm{tr}} + \frac{T_0}{T_e} - 1 \tag{A6}$$

where

$$H_{\mathrm{tr}} = \frac{\delta^*_{\mathrm{tr}}}{\theta_i}$$

APPENDIX B

DERIVATION OF RELATIONS BETWEEN SHAPE FACTORS

(a) Crocco Temperature Distribution

From Eq. (8) we have

$$\frac{T_s}{T_0} = a + b\,\frac{u}{u_e} = a + b\,\frac{U}{U_e}$$

where, from boundary conditions,

$$a = \frac{T_w}{T_0} \qquad b = 1 - \frac{T_w}{T_0} = 1 - a$$

Then, in Eq. (A5),

$$\delta^*_{\text{tr}} = \int_0^\Delta \left[a + (1 - a) \frac{U}{U_e} - \frac{U}{U_e} \right] dY$$

$$= \int_0^\Delta a \left(1 - \frac{U}{U_e}\right) dY$$

or

$$\delta^*_{\text{tr}} = a\delta^*_i = \frac{T_w}{T_0}\delta^*_i$$

where

$$\delta^*_i = \int_0^\Delta \left(1 - \frac{U}{U_e}\right) dY \qquad \text{as usual}$$

Now, from Eq. (6)

$$H_{\text{tr}} = \frac{\delta^*_{\text{tr}}}{\theta_i} = \frac{T_w}{T_0}\frac{\delta^*_i}{\theta_i} = \frac{T_w}{T_0}H_i \tag{B1}$$

Then, from Eq. (A6)

$$H = \frac{T_w}{T_e}H_i + \frac{T_0}{T_e} - 1 \tag{B2}$$

(b) van Driest Temperature Distribution

From Eq. (9) we have

$$\frac{T_s}{T_0} = (a + b)\frac{u}{u_e} + c\left(\frac{u}{u_e}\right)^2 = (a + b)\frac{U}{U_e} + c\left(\frac{U}{U_e}\right)^2$$

where the coefficients are defined as

$$a = \frac{T_w}{T_0} \qquad b = \frac{T_{\text{aw}}}{T_0} - \frac{T_w}{T_0} \qquad c = 1 - \frac{T_{\text{aw}}}{T_0}$$

and thus

$$b = \frac{T_{\text{aw}}}{T_0} - a = 1 - c - a$$

Then, in Eq. (A5),

$$\delta^*_{\text{tr}} = \int_0^\Delta \left[a + \frac{U}{U_e}(1 - c - a) + c\left(\frac{U}{U_e}\right)^2 - \frac{U}{U_e} \right] dY$$

$$= \int_0^\Delta \left[a\left(1 - \frac{U}{U_e}\right) - c\,\frac{U}{U_e}\left(1 - \frac{U}{U_e}\right)\right] dY$$

So

$$\delta^*_{\text{tr}} = a\delta^*_i - c\,\theta_i$$

Then, from Eq. (6)

$$H_{\text{tr}} = aH_i - c = \frac{T_w}{T_0}H_i + \frac{T_{\text{aw}}}{T_0} - 1 \tag{B3}$$

and from Eq. (A6)

$$H = \frac{T_w}{T_e}H_i + \frac{T_{\text{aw}}}{T_e} - 1 \tag{B4}$$

APPENDIX C

DEVELOPMENT OF MOMENTUM AND AUXILIARY EQUATIONS

(a) Momentum Equation

From Eq. (7)

$$\frac{d\theta_i}{dx} + \frac{\theta_i}{M_e}(2 + H_{\text{tr}})\frac{dM_e}{dx} = \frac{C_f}{2}\left(\frac{T_e}{T_0}\right)^3$$

or

$$\frac{d\theta_i}{dX} + \frac{\theta_i}{M_e}(2 + H_{\text{tr}})\frac{dM_e}{dX} = \frac{dx}{dX}\frac{C_f}{2}\left(\frac{T_e}{T_0}\right)^3 = \frac{C_f}{2}\,i$$

by transforming the longitudinal coordinate x. Thus

$$\frac{dX}{dx} = \frac{\dfrac{C_f}{2}\left(\dfrac{T_e}{T_0}\right)^3}{\dfrac{C_f}{2}\,i}$$

where

$$\frac{C_f}{2}i = 0.123\, e^{-1.561 H_i} \left(\frac{U_e \theta_i \rho_0}{\mu_0}\right)^{-0.268}$$

Then, using Eq. (3),

$$\frac{dX}{dx} = \frac{T_e}{T_r}\left(\frac{T_r}{T_0}\right)^{0.402}\left(\frac{T_0 + 198}{T_r + 198}\right)^{0.268}\left(\frac{T_e}{T_0}\right)^{3}$$

or

$$\frac{dX}{dx} = \frac{T_e}{T_r}\left(\frac{T_e}{T_0}\right)^{3}\left(\frac{\mu_r}{\mu_0}\right)^{0.268} \qquad \text{(C1)}$$

(b) Head's Auxiliary Equation

From Eq. (10)

$$\frac{d}{dX}(\Delta - \Delta^*) = F - \frac{(\Delta - \Delta^*)}{M_e}\frac{dM_e}{dX}$$

but

$$H_{\Delta-\Delta^*} = \frac{\Delta - \Delta^*}{\theta_i}$$

so

$$\frac{d}{dX}(H_{\Delta-\Delta^*}\theta_i) = F - \frac{H_{\Delta-\Delta^*}\theta_i}{M_e}\frac{dM_e}{dX}$$

or

$$\frac{d}{dx}(H_{\Delta-\Delta^*}\theta_i) = F\frac{dX}{dx} - \frac{\theta_i H_{\Delta-\Delta^*}}{M_e}\frac{dM_e}{dx}$$

Then

$$\theta_i \frac{dH_{\Delta-\Delta^*}}{dx} = F\frac{dX}{dx} - \frac{\theta_i H_{\Delta-\Delta^*}}{M_e}\frac{dM_e}{dx} - H_{\Delta-\Delta^*}\frac{d\theta_i}{dx}$$

or

$$\frac{dH_{\Delta-\Delta^*}}{dx} = \frac{F}{\theta_i}\frac{dX}{dx} - \frac{H_{\Delta-\Delta^*}}{M_e}\frac{dM_e}{dx} - \frac{H_{\Delta-\Delta^*}}{\theta_i}\frac{d\theta_i}{dx}$$

but

$$\frac{dH_{\Delta-\Delta^*}}{dx} = \frac{\partial H_{\Delta-\Delta^*}}{\partial H_i}\frac{dH_i}{dx}$$

and, from Eq. (11)

$$\frac{\partial H_{\Delta-\Delta^*}}{\partial H_i} = -\,4.17\ (H_i - 0.7)^{-3.715}$$

so

$$\frac{dH_i}{dx} = -\frac{(H_i - 0.7)^{3.715}}{4.17}\left(\frac{F}{\theta_i}\frac{dX}{dx} - \frac{H_{\Delta-\Delta^*}}{M_e}\frac{dM_e}{dx} - \frac{H_{\Delta-\Delta^*}}{\theta_i}\frac{d\theta_i}{dx}\right) \qquad \text{(C2)}$$

where dX/dx is given by Eq. (C1).

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ACKNOWLEDGEMENT

The author wishes to express his thanks to Professors S. Molder and B. G. Newman of McGill University for their advice and assistance in the analysis of the problem and the preparation of this paper. He also wishes to acknowledge the assistance of Mr. S. Kroll in preparing the initial draft of the paper. The financial assistance of the Defence Research Board of Canada, Grant No. 9550-06, is gratefully acknowledged.

COMMENTARY

D. G. CLARK (*Cambridge University, Engineering Laboratory, Cambridge, England*): Mr. Standen and delegates may be interested to know that further work on Head's method for the calculation of H has now been done at Cambridge by Dr. B. G. J. Thompson.

Thompson first examined existing methods for calculating H, using published experimental data for comparison. He found that the entrainment method gave a clear improvement over others in almost all cases, and, in addition, some accepted methods failed completely in certain circumstances.

Thompson has since modified Head's methods to give improved results in certain very exacting cases, and is now extending it to include the effects of distributed suction or blowing. As pointed out in the original paper, the concept of mass entrainment is well-suited to this more general application.

In the above connection, Thompson has also developed a new family of velocity profiles and an associated skin friction law. These too appear to be in good agreement with available experimental data.

It is hoped to publish these developments shortly and it is to be hoped that they also will be suitable for use with a transformation for application to the supersonic case.